pressure ratio of expansion = 0.15, and ratio of specific heats = 1.33.

The loss in nozzle efficiency (loss of exit momentum) is plotted in Fig. 3 as a function of the Mach number of one of the adjacent streams (designated stream 1). In this computation the mass flow is assumed to be divided equally between the two adjacent streams.

Conclusions

It would appear that the unrecoverable loss in thrust resulting from nonuniform velocity at the nozzle entrance is not likely to exceed about 1%, unless large variations in velocity are encountered. However, some reservations should be noted. For this example, the loss in thrust is about four times as great as the loss in nozzle efficiency. Also, the thrust in this example is quite high. In more marginal engines, a small loss in exit momentum could be much more significant. In addition, it is important to recall that it has been assumed here that the nonuniform inlet conditions are known and have been taken into account in the nozzle design. If the design is faulty, then larger losses are possible, and may be inevitable in a flight environment involving changes of angle of attack, altitude, and Mach number.

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Multimode Control of Axial Compressors via Stability-Based Switching Controllers

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I. Introduction

A FUNDAMENTAL development in compression system modeling for low-speed axial compressors is the Moore–Greitzer model given in Ref. 1. Specifically, utilizing a one-mode expansion of the disturbance velocity potential in the compression system and assuming a cubic characteristic for the compressor performance map, the authors in Ref. 1 developed a low-order three-state nonlinear model involving the mean flow in the compressor, the pressure rise, and the amplitude of the rotating stall. However, a shortcoming of the low-order three-state Moore–Greitzer model and, as a consequence, the control design methodologies based on the model, is that only a one-mode expansion of the disturbance velocity potential in the compression system is considered. Because the second and higher-order disturbance velocity potential harmonics strongly interact with the first harmonic during stall inception, they must be accounted for in the control-system design process. A notable exception to the low-order three-state model is given in Ref. 2, where a discrete Fourier transform formulation is used to obtain a distributed nonlinear model for axial compression systems.

Using Lyapunov stability theory, in this paper a novel switching nonlinear globally stabilizing control law for a multimode axial flow compressor model, based on equilibria-dependent Lyapunov functions with converging domains of attraction, is developed. The locus of equilibrium points on which the equilibria-dependent, or instantaneous, Lyapunov functions are predicated, is characterized by the axisymmetric pressure-flow equilibria of the compression system. The proposed switching nonlinear controller is directly applicable to compression systems with actuator amplitude and rate saturation constraints while providing a guaranteed domain of attraction.

II. Fluid Dynamic Equations for Axial Compression Systems

In this section, we extend the single-mode Moore–Greitzer model for rotating stall and surge in axial flow compressors given in Ref. 1 to the multimode case. Specifically, we consider the basic compression system consisting of an inlet duct, compressor, outlet duct, plenum, and control throttle.

From an analysis of the flow in the entrance duct, the inlet guide vane entrance, the compressor, and the exit duct, we obtain the expression for the pressure rise between the upstream reservoir and the exit duct discharge given by[1]

$$\Psi = \psi_C(\phi) - l_C \frac{d\Phi}{d\xi} - m \left.\frac{\partial\tilde{\varphi}}{\partial\xi}\right|_{\eta=0} - \frac{1}{2a}\left[2\frac{\partial^2\tilde{\varphi}}{\partial\eta\partial\xi} + \frac{\partial^2\tilde{\varphi}}{\partial\eta\partial\theta}\right]_{\eta=0} \tag{1}$$

where η, θ, and ξ are the nondimensional axial, circumferential, and time coordinates, respectively; m is a parameter such that $m = 1$ for a very short exit duct and $m = 2$ otherwise; $\phi(\xi, \theta)$ is the axial flow coefficient at $\eta = 0$; $\Phi(\xi)$ is the circumferential-averaged component of $\phi(\xi, \theta)$; and $\tilde{\varphi}(\xi, \theta, \eta)$ is the perturbation velocity potential. Furthermore, $\Psi \triangleq (p_S - p_T)/\rho U^2$, $l_C \triangleq l_E + l_I + (1/a)$, are, respectively, the rise between the static pressure, p_S, and the total pressure, p_T, normalized with respect to the dynamic pressure, ρU^2, and the effective flow-passage length through the compressor [pseudolength $(1/a)$] the inlet and exit ducts (length l_I and l_E, respectively), all measured in radii of the compressor wheel. Finally, $\psi_C(\phi)$ is the quasisteady axisymmetric compressor characteristic pressure-flow map, and represents the compressor performance in the case where the flow through the compressor is circumferentially uniform and steady, even in a stalled condition.

Assuming that the velocity field is unperturbed at the entrance, we obtain that the perturbation velocity potential satisfies $\nabla^2\tilde{\varphi} = 0$ with

$$\left.\frac{\partial\tilde{\varphi}}{\partial\eta}\right|_{\eta=-l_I} = 0$$

whose solution can be written as

$$\tilde{\varphi}(\xi, \theta, \eta) = \sum_{k=1}^{\infty} [a_k(\xi)\sin(k\theta) + b_k(\xi)\cos(k\theta)] \times \frac{\cosh[k(\eta + l_I)]}{k\sinh(kl_I)}, \qquad \eta \le 0 \tag{2}$$

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Now, substituting Eq. (2) into Eq. (1), and computing the circumferential mean of the resulting equation, we obtain

$$l_C \frac{d\Phi}{d\xi} = -\Psi + \frac{1}{2\pi}\int_0^{2\pi} \psi_C(\phi)\, d\theta \tag{3}$$

which relates the change of mass flow through the compressor to the total pressure rise.

Because the plenum dimensions are assumed large as compared with the compressor-duct dimensions, the pressure in the plenum is spatially uniform. Using continuity arguments, it follows that

$$l_C \frac{d\Psi}{d\xi} = \frac{1}{4B^2}[\Phi(\xi) - \Phi_T(\xi)] \tag{4}$$

where $\Phi_T(\xi)$ is the nondimensional mass flow through the throttle, and B is a nondimensional compliance parameter and is a function of the rotor speed and plenum size.

Next, assuming that the throttle discharges to an infinite reservoir with pressure p_T, it follows that the pressure difference $p_S - p_T$ must balance both the throttle pressure loss and the net difference in pressure caused by the flow acceleration through the throttle duct. Hence, $\Psi(\xi) = F_T(\Phi_T)$, where $F_T(\Phi_T)$ represents the throttle pressure loss. Here, as in Ref. 1, we consider a quadratic throttle characteristic given by $F_T(\Phi_T) = \frac{1}{2}K_T\Phi_T^2$, where K_T is a constant throttle coefficient. In this case, it follows that $\Phi_T = F_T^{-1}(\Psi) = \gamma\sqrt{\Psi}$, where the parameter γ is proportional to the throttle opening. Finally, substituting Φ_T into Eq. (4) yields

$$l_C \frac{d\Psi}{d\xi} = \frac{1}{4B^2}[\Phi(\xi) - F_T^{-1}(\Psi)] \tag{5}$$

III. Multimode State-Space Model

In this section, we develop a multimode state-space model for the axial compression system addressed in Sec. II. The governing system flow equations for the axial flow compressor model are given by Eqs. (1), (3), and (5). Using Eq. (3), Eq. (1) can be rewritten as

$$\psi_C(\phi) - \frac{1}{2\pi}\int_0^{2\pi}\psi_C(\phi)\, d\theta = m\left.\frac{\partial\tilde{\varphi}}{\partial\xi}\right|_{\eta=0} + \frac{1}{2a}\left[2\frac{\partial^2\tilde{\varphi}}{\partial\eta\partial\xi} + \frac{\partial^2\tilde{\varphi}}{\partial\eta\partial\theta}\right]_{\eta=0} \tag{6}$$

Substituting Eq. (2) for $\tilde{\varphi}(\xi, \theta, \eta)$ into Eq. (6), we obtain

$$\psi_C(\phi) - \frac{1}{2\pi}\int_0^{2\pi}\psi_C(\phi)\, d\theta = \sum_{k=1}^{\infty}\left[\left(\alpha_k\frac{da_k}{d\xi} - \beta_k b_k\right)\sin(k\theta) + \left(\alpha_k\frac{db_k}{d\xi} + \beta_k a_k\right)\cos(k\theta)\right] \tag{7}$$

where

$$\alpha_k \triangleq \frac{m}{k}\frac{\cosh(kl_I)}{\sinh(kl_I)} + \frac{1}{a}, \qquad \beta_k \triangleq \frac{k}{2a}$$

$$\phi = \Phi + \left.\frac{\partial\tilde{\varphi}}{\partial\eta}\right|_{\eta=0} = \Phi + \sum_{k=1}^{\infty}[a_k(\xi)\sin(k\theta) + b_k(\xi)\cos(k\theta)]$$

Applying a Galerkin formulation to Eq. (7), and using $\sin(k\theta)$ and $\cos(k\theta)$, $k = 1, \ldots, n$, as projection functions, we obtain the n-mode model given by

$$\alpha_k\frac{da_k}{d\xi} - \beta_k b_k = \psi_{c,k}^{\sin} \triangleq \frac{1}{\pi}\int_0^{2\pi}\psi_C(\phi)\sin(k\theta)\, d\theta \tag{8}$$

$$\alpha_k\frac{db_k}{d\xi} + \beta_k a_k = \psi_{c,k}^{\cos} \triangleq \frac{1}{\pi}\int_0^{2\pi}\psi_C(\phi)\cos(k\theta)\, d\theta \tag{9}$$

Next, combining Eqs. (8) and (9), and rewriting Eqs. (3) and (5), we obtain

$$\frac{d\hat{x}}{d\xi} = \hat{A}\hat{x} + \hat{f}(\hat{x}, \Phi), \qquad \frac{d\Phi}{d\xi} = \frac{1}{l_C}(\psi_{c,0} - \Psi), \qquad \frac{d\Psi}{d\xi} = \frac{1}{4B^2 l_C}[\Phi - \gamma\sqrt{\Psi}] \tag{10}$$

where

$$\psi_{c,0} \triangleq \frac{1}{2\pi}\int_0^{2\pi}\psi_C(\phi)\, d\theta, \qquad \hat{x} \triangleq [a_1\ b_1 \ldots a_n\ b_n]^T$$

$$\hat{f}(\Phi, \hat{x}) \triangleq D_\alpha^{-1}[\psi_{c,1}^{\sin}\ \psi_{c,1}^{\cos} \ldots \psi_{c,n}^{\sin}\ \psi_{c,n}^{\cos}]^T$$

$$\hat{A} \triangleq D_\alpha^{-1}\text{block-diag}_{k=1,\ldots,n}\begin{bmatrix} 0 & \beta_k \\ -\beta_k & 0\end{bmatrix}$$

$$D_\alpha \triangleq \text{block-diag}_{k=1,\ldots,n}\begin{bmatrix} \alpha_k & 0 \\ 0 & \alpha_k\end{bmatrix}$$

Equation (10) gives a $2n + 2$ order state-space model for the compression problem with state variables $\hat{x}$, Φ, and Ψ, and control variable γ.

The quasisteady, axisymmetric compressor characteristic map $\psi_C(\phi)$, considered in the literature,[1] is given by the cubic function:

$$\psi_C(\phi) = \psi_{C_0} + H\left[1 + \frac{3}{2}\left(\frac{\phi}{W} - 1\right) - \frac{1}{2}\left(\frac{\phi}{W} - 1\right)^3\right]$$

where ψ_{C_0}, H and W are parameters that can be used to shape the compressor characteristic map. In the case where $n = 1$, Eq. (10) collapses to the standard three-state Moore–Greitzer model.[1]

IV. Finite Element State-Space Model

Because the state-space model, given in Sec. III, requires the computation of $\psi_{c,0}$, $\psi_{c,k}^{\sin}$, and $\psi_{c,k}^{\cos}$, $k = 1, \ldots, n$, which involve integrals of transcendental functions, in this section we give an alternative state-space basis that eliminates this complexity. Our state transformation only involves the state variables Φ, a_k, and b_k, $k = 1, \ldots, n$, so that we need only consider the truncated state vector $x_t \triangleq [\hat{x}\ \Phi]^T$. Specifically, we consider $n_t \triangleq 2n + 1$ flow state variables given by

$$\phi_i \triangleq \Phi + \sum_{k=1}^{n}[a_k(\xi)\sin(k\theta_i) + b_k(\xi)\cos(k\theta_i)]$$

where

$$\theta_i \triangleq \frac{2\pi i}{n_t}, \qquad i = 1, \ldots, n_t$$

Hence, define the flow state vector $\hat{\phi} \triangleq [\phi_1\ \phi_2 \ldots \phi_{n_t}]^T$, so that $\hat{\phi} = Sx_t$, where

$$S = \begin{bmatrix} \sin(\theta_1) & \cos(\theta_1) & \cdots & \sin(n\theta_1) & \cos(n\theta_1) & 1 \\ \sin(\theta_2) & \cos(\theta_2) & \cdots & \sin(n\theta_2) & \cos(n\theta_2) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \sin(\theta_{n_t}) & \cos(\theta_{n_t}) & \cdots & \sin(n\theta_{n_t}) & \cos(n\theta_{n_t}) & 1 \end{bmatrix} \tag{11}$$

Now, applying the preceding change of variables, the state-space description for the dynamics of $\hat{\phi}$ is given by

$$\frac{d\hat{\phi}}{d\xi} = A_s\hat{\phi} + D_s^{-1}\left(\begin{bmatrix}\hat{\psi}_C(\theta_1) \\ \vdots \\ \hat{\psi}_C(\theta_{n_t})\end{bmatrix} - \begin{bmatrix}\Psi \\ \vdots \\ \Psi\end{bmatrix}\right) \tag{12}$$

where

$$A_s \triangleq S \begin{bmatrix} \hat{A} & 0_{2n\times 1} \\ 0_{1\times 2n} & 0 \end{bmatrix} S^{-1}, \qquad D_s \triangleq S \begin{bmatrix} D_\alpha & 0_{2n\times 1} \\ 0_{1\times 2n} & l_C \end{bmatrix} S^{-1}$$

$$\hat{\psi}_C(\theta_i) = \psi_{c,0} + \sum_{k=1}^{n} [\psi_{c,k}^{\sin} \sin(k\theta_i) + \psi_{c,k}^{\cos} \cos(k\theta_i)]$$

It is interesting to note that $\hat{\psi}_C(\theta_i)$ is a truncated Fourier expansion of $\psi_C(\phi_i)$, and therefore, we introduce an approximation in Eq. (12) by replacing $\hat{\psi}_C(\theta_i)$ with $\psi_C(\phi_i)$. Hence, including the equation for the state variable Ψ, the new state-space model becomes

$$\frac{d\hat{\phi}}{d\xi} = A_s\hat{\phi} + D_s^{-1}\psi_C(\hat{\phi}) - e\Psi, \qquad \frac{d\Psi}{d\xi} = \frac{1}{4B^2 l_C}\left(\frac{e^T\hat{\phi}}{n_t} - \gamma\sqrt{\Psi}\right) \tag{13}$$

where

$$e \triangleq [1 \;\; 1 \;\; \ldots \;\; 1]^T \in \mathbb{R}^{n_t\times 1}$$

$$\psi_C(\hat{\phi}) \triangleq [\psi_C(\phi_1) \; \ldots \; \psi_C(\phi_{n_t})]^T$$

Note that $e^T\hat{\phi}/n_t = \Phi$, A_s is skew symmetric, and D_s is nonsingular with positive eigenvalues. Furthermore, e is an eigenvector of A_s and D_s associated with the eigenvalues 0 and l_C, respectively.

V. Stability-Based Switching Nonlinear Controllers

In this section, we develop a globally stabilizing control strategy for controlling the multimode compression system [Eq. (13)]. Specifically, using Lyapunov stability theory, a novel switching nonlinear globally stabilizing control law based on equilibria-dependent or instantaneous (with respect to given equilibrium) Lyapunov functions, with a converging domain of attractions, is developed. The locus of equilibrium points on which the instantaneous Lyapunov functions are predicted is characterized by the axisymmetric stable pressure-flow equilibrium branch of Eq. (13) for a continuum of mass flow through the throttle. For this development, define the shifted variables $\hat{\phi}_s \triangleq (1/W)(\hat{\phi} - 2We)$ and $\Psi_s \triangleq (1/H)(\Psi - \psi_{C_0} - 2H)$, so that the maximum pressure point on the compressor characteristic pressure-flow map is translated to the origin. In this case, the translated nonlinear system is given by

$$\dot{\hat{\phi}}_s = A\hat{\phi}_s + P^{-1}\psi_{sC}(\hat{\phi}_s) - e\Psi_s, \qquad \dot{\Psi}_s = (1/\beta^2)[(e^T\hat{\phi}_s/n_t) - u] \tag{14}$$

where

$$A \triangleq \frac{Wl_C}{H}A_s, \qquad P \triangleq \frac{1}{l_C}D_s, \qquad \beta \triangleq \frac{2BH}{W}, \qquad u \triangleq \Phi_{sT} = \frac{\Phi_T}{W} - 2$$

$$\psi_{sC}(\hat{\phi}_s) \triangleq [\psi_{sC}(\phi_{s1}) \ldots \psi_{sC}(\phi_{sn_t})]^T, \qquad \psi_{sC}(\phi_{si}) = -\tfrac{3}{2}\phi_{si}^2 - \tfrac{1}{2}\phi_{si}^3$$

and ($\dot{\ }$) represents differentiation with respect to the nondimensional time $\xi_s \triangleq (H/Wl_C)\xi$. Note that for $u = \lambda$, where $\lambda \geq 0$, Eq. (14) has an equilibrium point at $(\hat{\phi}_{s\lambda}, \Psi_{s\lambda})$, where $\hat{\phi}_{s\lambda} \triangleq \lambda e$ and $\Psi_{s\lambda} \triangleq \psi_{sC}(\lambda) = -\tfrac{3}{2}\lambda^2 - \tfrac{1}{2}\lambda^3$.

Next, we show that for $\lambda > 0$ there exists a control law such that the equilibrium point $(\hat{\phi}_{s\lambda}, \Psi_{s\lambda})$ of Eq. (14) is locally asymptotically stable with a domain of attraction $\mathcal{D}_\lambda$. Consider the equilibrium-dependent Lyapunov function candidate:

$$V_\lambda(\hat{\phi}_s, \Psi_s) = \tfrac{1}{2}\beta^2[\Psi_s - \Psi_{s\lambda}]^2 + (1/2n_t)(\hat{\phi}_s - \hat{\phi}_{s\lambda})^T P(\hat{\phi}_s - \hat{\phi}_{s\lambda}) \tag{15}$$

with Lyapunov derivative

$$\dot{V}_\lambda(\hat{\phi}_s, \Psi_s) = -(u - \lambda)[\Psi_s - \psi_{sC}(\lambda)] - \frac{1}{n_t}\sum_{i=1}^{n_t}(\phi_{si} - \lambda)[\psi_{sC}(\lambda) - \psi_{sC}(\phi_{si})] \tag{16}$$

Substituting $\psi_{sC}(\phi_{si})$ into Eq. (16), and choosing $u(\hat{\phi}_s, \Psi_s) = u_\lambda(\hat{\phi}_s, \Psi_s) \triangleq \lambda + h_\lambda(\hat{\phi}_s, \Psi_s)$, where $h_\lambda:\mathbb{R}^{n_t} \times \mathbb{R} \to \mathbb{R}$ is such that $h_\lambda(\hat{\phi}_{s\lambda}, \Psi_{s\lambda}) = 0$, we obtain

$$\dot{V}_\lambda(\hat{\phi}_s, \Psi_s) = -h_\lambda(\hat{\phi}_s, \Psi_s)[\Psi_s - \psi_{sC}(\lambda)] - \frac{1}{2n_t}\sum_{i=1}^{n_t}(\phi_{si} - \lambda)^2[(\phi_{si})^2 + (\lambda + 3)\phi_{si} + \lambda(\lambda + 3)] \tag{17}$$

Now, a sufficient condition guaranteeing that $\dot{V}_\lambda(\hat{\phi}_s, \Psi_s) < 0$, $(\hat{\phi}_s, \Psi_s) \in \mathcal{D}_\lambda\backslash(\hat{\phi}_{s\lambda}, \Psi_{s\lambda})$, is given by

$$\phi_{si}^2 + (\lambda + 3)\phi_{si} + \lambda(\lambda + 3) > 0, \qquad i = 1, \ldots, n_t \tag{18}$$

$$h_\lambda(\hat{\phi}_s, \Psi_s)[\Psi_s - \psi_{sC}(\lambda)] > 0, \qquad (\hat{\phi}_s, \Psi_s) \neq (\hat{\phi}_{s\lambda}, \Psi_{s\lambda}) \tag{19}$$

Note that Eq. (18) holds if $\lambda > 1$ or

$$\phi_{si} > -d_\lambda, \qquad d_\lambda \triangleq \tfrac{1}{2}[(\lambda + 3) - \sqrt{3(\lambda + 3)(1 - \lambda)}] \qquad i = 1, \ldots, n_t, \qquad 0 \leq \lambda \leq 1 \tag{20}$$

whereas a particular choice of $h_\lambda(\cdot, \cdot)$, satisfying Eq. (19), is given by $h_\lambda(\hat{\phi}_s, \Psi_s) \triangleq w[\Psi_s - \psi_{sC}(\lambda)]p(\hat{\phi}_s - \lambda e)$, where $w:\mathbb{R} \to \mathbb{R}$ is such that $xw(x) > 0$, $x \neq 0$, and $p:\mathbb{R}^{n_t} \to \mathbb{R}$ is positive definite. In this case, $\dot{V}_\lambda(\hat{\phi}_s, \Psi_s) < 0$, $(\hat{\phi}_s, \Psi_s) \in \mathcal{D}_\lambda\backslash(\hat{\phi}_{s\lambda}, \Psi_{s\lambda})$, and hence, the equilibrium point $(\hat{\phi}_{s\lambda}, \Psi_{s\lambda})$ of Eq. (14) is locally asymptotically stable for $\lambda \in [0, 1]$ and globally asymptotically stable for $\lambda > 1$.

Next, with $u(\hat{\phi}_s, \Psi_s) = u_\lambda(\hat{\phi}_s, \Psi_s)$, we provide an estimate of the domain of attraction for Eq. (14). In particular, $\mathcal{D}_\lambda = \mathbb{R}^{n_t} \times \mathbb{R}$, $\lambda > 1$, and $\mathcal{D}_\lambda = [(\hat{\phi}_s, \Psi_s):V_\lambda(\hat{\phi}_s, \Psi_s) \leq c_\lambda]$, $0 \leq \lambda \leq 1$, where

$$c_\lambda \triangleq \frac{\mu}{2n_t}(d_\lambda + \lambda)^2, \qquad \mu \triangleq \frac{1}{l_C}\min\left(l_C, \frac{m}{n} + \frac{1}{a}\right)$$

is a subset of the domain of attraction of Eq. (14). The Lyapunov contour surfaces $V_\lambda(\hat{\phi}_s, \Psi_s) = c_\lambda$ are defined such that the intersection of the boundary of $\mathcal{D}_\lambda$ with the plane $\Psi_s = \Psi_{s\lambda}$ is a closed surface contained in the region $(\hat{\phi}_s:\phi_{si} > -d_\lambda$, $i = 1, \ldots, n_t)$, so that $\dot{V}_\lambda(\hat{\phi}_s, \Psi_s) < 0$ for all $(\hat{\phi}_s, \Psi_s) \in \mathcal{D}_\lambda\backslash(\hat{\phi}_{s\lambda}, \Psi_{s\lambda})$. Note that, because $c_\lambda \geq 0$ and $V_\lambda(\hat{\phi}_s, \Psi_s)$ is continuous and radially unbounded, $\mathcal{D}_\lambda$ is a compact set for $\lambda \in [0, 1]$, which further implies that $\mathcal{D}_\lambda$ is a positive invariant set. Finally, because $\mathcal{D}_\lambda$ is not empty for all $\lambda > 0$, there exists λ_1, $\lambda_2 > 0$, such that $\lambda_2 < \lambda_1$ and $(\hat{\phi}_{s\lambda_1}, \Psi_{s\lambda_1}) \in \mathring{\mathcal{D}}_{\lambda_2}$, where $\mathring{\mathcal{D}}_\lambda$ denotes the interior of $\mathcal{D}_\lambda$, that is, $\mathring{\mathcal{D}}_\lambda \triangleq \{(\hat{\phi}_s, \Psi_s):V_\lambda(\hat{\phi}_s, \Psi_s) < c_\lambda\}$.

Using the properties of $\mathcal{D}_\lambda$, we now present the following globally stabilizing switching nonlinear control strategy. Let $\{\lambda_1, \ldots, \lambda_q\}$ be such that $\lambda_1 > 1$, $\lambda_1 > \lambda_2 > \cdots > \lambda_q > 0$, and $(\hat{\phi}_{s\lambda_i}, \Psi_{s\lambda_i}) \in \mathring{\mathcal{D}}_{\lambda_{i+1}}$, $i \in \{1, \ldots, q - 1\}$. Furthermore, define

$$\lambda(\hat{\phi}_s, \Psi_s) \triangleq \min_{i=1,\ldots,q}\{\lambda_i:(\hat{\phi}_s, \Psi_s) \in \mathcal{D}_{\lambda_i}\} \tag{21}$$

Because for each $i \in \{1, \ldots, q - 1\}$, $(\hat{\phi}_{s\lambda_i}, \Psi_{s\lambda_i}) \in \mathring{\mathcal{D}}_{\lambda_{i+1}}$ it follows that $\lambda(\hat{\phi}_s, \Psi_s)$ is a nonincreasing function of time along the state trajectories of Eq. (14). Hence, with $u(\hat{\phi}_s, \Psi_s) =$

$u_{\lambda(\hat{\phi}_s,\Psi_s)}(\hat{\phi}_s, \Psi_s)$ the solution $(\hat{\phi}_s, \Psi_s)$ of Eq. (14) approaches $\mathfrak{D}_{\lambda_q}$ as $t \to \infty$. Now, choosing $\lambda_q \to 0^+$, it follows that $(\hat{\phi}_s, \Psi_s) \to (0, 0)$ as $t \to \infty$, which shows that $u(\hat{\phi}_s, \Psi_s) = u_{\lambda(\hat{\phi}_s,\Psi_s)}(\hat{\phi}_s, \Psi_s)$, globally stabilizes the equilibrium point (0, 0) of Eq. (14).

Note that the control law $u(\hat{\phi}_s, \Psi_s)$ is a stability-based switching controller, and hence, discontinuous. Next, we present a modification to Eq. (21), so that the resulting control law is continuous modulo at most one discontinuity. For this development, define the compact set $\mathfrak{D} \triangleq \cup_{0\le\lambda\le1}\mathfrak{D}_\lambda$, consisting of the union of the compact sets, $\mathfrak{D}_\lambda$, $\lambda \in [0, 1]$. Next, if $(\hat{\phi}_s(0), \Psi_s(0)) \notin \mathfrak{D}$, setting $u(\hat{\phi}_s, \Psi_s) = u_{1+\varepsilon}(\hat{\phi}_s, \Psi_s)$, the state trajectories $(\hat{\phi}_s(t), \Psi_s(t))$, $t \ge 0$, will approach the globally asymptotically stable equilibrium point $(\hat{\phi}_{s(1+\varepsilon)}, \Psi_{s(1+\varepsilon)})$. In particular, if ε is such that $(\hat{\phi}_{s(1+\varepsilon)}, \Psi_{s(1+\varepsilon)}) \in \mathring{\mathfrak{D}}_1$, then there exists $t > 0$ such that $(\hat{\phi}_s(t), \Psi_s(t)) \in \mathring{\mathfrak{D}}_1 \subset \mathfrak{D}$. Now, let $\bar{t}$ be such that $(\hat{\phi}_s(\bar{t}), \Psi_s(\bar{t})) \in \mathfrak{D}$, and define

$$\lambda(\hat{\phi}_s, \Psi_s) \triangleq (1 + \varepsilon)\inf_{\lambda>0}\{\lambda:(\hat{\phi}_s, \Psi_s) \in \mathfrak{D}_\lambda\} \tag{22}$$

and $\bar{\lambda} \triangleq \lambda(\hat{\phi}_s(\bar{t}), \Psi_s(\bar{t}))$. From the definition of $\lambda(\cdot, \cdot)$ it follows that $(\hat{\phi}_s(\bar{t}), \Psi_s(\bar{t}))$ is on $\partial\mathfrak{D}_{\bar{\lambda}}$, where $\partial\mathfrak{D}_\lambda$ denotes the boundary of $\mathfrak{D}_\lambda$, i.e., $\partial\mathfrak{D}_\lambda \triangleq \{(\hat{\phi}_s, \Psi_s):V_\lambda(\hat{\phi}_s, \Psi_s) = c_\lambda\}$. Furthermore, because $\dot{V}_{\bar{\lambda}}(\hat{\phi}_s(\bar{t}), \Psi_s(\bar{t})) < 0$, $(\hat{\phi}_s, \Psi_s) \in \mathfrak{D}_{\bar{\lambda}}\backslash(\hat{\phi}_{s\bar{\lambda}}, \Psi_{s\bar{\lambda}})$, it follows that there exists $\delta > 0$, such that $V_{\bar{\lambda}}(\hat{\phi}_s(t), \Psi_s(t)) < c_{\bar{\lambda}}$, $t \in [\bar{t}, \bar{t} + \delta)$. Hence, $\lambda(\hat{\phi}_s(t), \Psi_s(t)) < \lambda(\hat{\phi}_s(\bar{t}), \Psi_s(\bar{t}))$, $t \in [\bar{t}, \bar{t} + \delta)$. Because $\bar{t}$ was chosen arbitrarily, it follows that if $(\hat{\phi}_s(0), \Psi_s(0)) \in \mathfrak{D}$, then $\lambda(\hat{\phi}_s(t), \Psi_s(t))$, $t \ge 0$, is monotonically decreasing.

Now, with $u(\hat{\phi}_s, \Psi_s) = u_{\lambda(\hat{\phi}_s,\Psi_s)}(\hat{\phi}_s, \Psi_s)$, define the Lyapunov function candidate

$$V(\hat{\phi}_s, \Psi_s) \triangleq V_{\lambda(\hat{\phi}_s,\Psi_s)}(\hat{\phi}_s, \Psi_s) = c_{\lambda(\hat{\phi}_s,\Psi_s)} = \frac{\mu}{2n_t}(d_{\lambda(\hat{\phi}_s,\Psi_s)} + \lambda(\hat{\phi}_s, \Psi_s))^2, \quad (\hat{\phi}_s, \Psi_s) \in \mathfrak{D} \tag{23}$$

with Lyapunov derivative

$$\dot{V}(\hat{\phi}_s, \Psi_s) = \frac{\mu}{n_t}(d_{\lambda(\hat{\phi}_s,\Psi_s)} + \lambda(\hat{\phi}_s, \Psi_s))\left(\frac{\mathrm{d}d_\lambda}{\mathrm{d}\lambda}(\hat{\phi}_s, \Psi_s) + 1\right) \times \dot{\lambda}(\hat{\phi}_s, \Psi_s) \tag{24}$$

Because $\dot{\lambda}(\hat{\phi}_s, \Psi_s) < 0$ for $(\hat{\phi}_s, \Psi_s) \in \mathfrak{D}$ and $(\mathrm{d}d_\lambda/\mathrm{d}\lambda) > 0$, it follows that $\dot{V}(\hat{\phi}_s, \Psi_s) < 0$ for $(\hat{\phi}_s, \Psi_s) \in \mathfrak{D}$, which proves local asymptotic stability of the origin.

Now, to construct a globally stabilizing controller, it need only be noted that $u(\hat{\phi}_s, \Psi_s) = u_{1+\varepsilon}(\hat{\phi}_s, \Psi_s)$, if $(\hat{\phi}_s(0), \Psi_s(0)) \notin \mathfrak{D}$ and $u(\hat{\phi}_s, \Psi_s) = u_{\lambda(\hat{\phi}_s,\Psi_s)}(\hat{\phi}_s, \Psi_s)$; otherwise, where $\lambda(\hat{\phi}_s, \Psi_s)$ is given by Eq. (22). However, this control law may be discontinuous at the boundary of $\mathfrak{D}$. Alternatively, a continuous control law that globally stabilizes the origin of the system can be obtained by setting $u(\hat{\phi}_s, \Psi_s) = u_{1+\varepsilon}(\hat{\phi}_s, \Psi_s)$, if $(\hat{\phi}_s(0), \Psi_s(0)) \notin \mathfrak{D}$, and letting the stage trajectories enter the domain $\mathfrak{D}_1 \subset \mathfrak{D}$ before switching the control law to $u(\hat{\phi}_s, \Psi_s) = u_{\lambda(\hat{\phi}_s,\Psi_s)}(\hat{\phi}_s, \Psi_s)$, where $\lambda(\hat{\phi}_s, \Psi_s)$ is given by Eq. (22).

The condition $V_\lambda(\hat{\phi}_s, \Psi_s) = c_\lambda$, which characterizes the boundary of $\mathfrak{D}_\lambda$, $\lambda \in [0, 1]$, depends on d_λ given by Eq. (20). This equation, however, is irrational and its solution is not easily computable. An alternative approach for updating λ on line can be obtained by noting that the condition $V_\lambda(\hat{\phi}_s, \Psi_s) = c_\lambda$ must be satisfied for all $t \ge 0$, and hence, its time derivative must also be satisfied for all $t \ge 0$. In particular, using Eq. (24), and noting that $\dot{V}(\hat{\phi}_s, \Psi_s) = \dot{V}_\lambda(\hat{\phi}_s, \Psi_s) + (\partial V_\lambda/\partial\lambda)\dot{\lambda}$, where $\dot{V}_\lambda(\hat{\phi}_s, \Psi_s)$ is given by Eq. (17), we obtain

$$\dot{\lambda} = \dot{V}_\lambda(\hat{\phi}_s, \Psi_s)\bigg/\left[\frac{\mu}{n_t}(d_\lambda + \lambda)\left(\frac{\mathrm{d}d_\lambda}{\mathrm{d}\lambda} + 1\right) - \lambda + \frac{e^T\hat{\phi}_s}{n_t} - 3\lambda\beta^2\left(\Psi_s + \frac{3}{2}\lambda^2 + \frac{1}{2}\lambda^3\right)\left(1 + \frac{1}{2}\lambda\right)\right] \tag{25}$$

with $\lambda(0) = \lambda_0$, such that $V_{\lambda_0}(\hat{\phi}_{s_0}, \Psi_{s_0}) = c_{s_0}$. Note that Eq. (25), along with $u(\hat{\phi}_s, \Psi_s) = u_\lambda(\hat{\phi}_s, \Psi_s)$, provides a nonlinear first-order dynamic compensator equivalent to the original condition, $V_\lambda(\hat{\phi}_s, \Psi_s) = c_\lambda$, which now needs only to be solved once to compute the initial condition λ_0. Note that the compensator dynamics given by Eq. (25) characterize the admissible rate of the compensator state λ such that the switching nonlinear controller guarantees that $(\hat{\phi}_s(t), \Psi_s(t)) \in \partial\mathfrak{D}_{\lambda(t)}$, $t \ge 0$.

Finally, because all control actuation devices are subject to amplitude and rate saturation constraints that lead to saturation nonlinearities, we discuss how the proposed switching nonlinear controller can be incorporated to address such practical limitations. Specifically, because the dynamic compensator state λ is proportional to the throttle opening (actuator), and because the dynamics given by Eq. (25) indirectly characterize the fastest admissible rate at which the control throttle can open while maintaining stability of the controlled system, it follows that, by constraining the rate at which the dynamics of λ can evolve on the equilibrium branch, the controller effectively places a rate constraint on the throttle opening. Mathematically, this corresponds to the case where the switching rate of the nonlinear controller is decreased, so that the trajectory $(\hat{\phi}_s(t), \Psi_s(t))$, $t \ge 0$, is allowed to enter $\mathfrak{D}_{\lambda(t)}$. Additionally, amplitude saturation constraints and state constraints can also be enforced by simply choosing $\lambda_{\max} < 1$, such that $\mathfrak{D}_{\max} \triangleq \cup_{0\le\lambda\le\lambda_{\max}}\mathfrak{D}_\lambda$ is contained in the region where the system is constrained to operate. In this case, the Lyapunov stability-based switching nonlinear controller provides a local stability guarantee with domain of attraction given by $\mathfrak{D}_{\max}$. Of course, in practice, it is sufficient to implement controllers with adequate domains of attraction and a priori saturation constraint guarantees, rather than implementing global controllers without realistic actuator limitations.

VI. Conclusions

A multimode state-space model for rotating stall and surge in axial flow compression systems that lends itself to the application of nonlinear control design was developed. Using Lyapunov stability theory, a nonlinear globally stabilizing control law based on a stability-based switching strategy with converging domains of attraction was developed.

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